The SU(5) Grand Unified Theory

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1 Introduction

There are many ways of combining the standard model SU(3)×SU(2)×U(1) symmetry into one single Lie group. These are called Grand Unified Theories (GUT). This report describes the simplest GUT, namely SU(5). The SU(5) has a number of appealing phenomenological implications, which will also be briefly discussed.

2 Standard model particles and symmetries

The standard model has a SU(3)×SU(2)×U(1) symmetry where the quarks are SU(3) triplets and the left-handed particles and right-handed antiparticles are SU(2) doublets. The remaining particles are singlets under both these symmetries. Looking at the first generation only, the right-handed antiparticles can be put into SU(2) doublets (the other generations can be treated in the same way):

$$\overline{\psi}^\dagger = \begin{pmatrix} \overline{u} \\ \overline{d} \end{pmatrix}, \qquad \overline{l}^\dagger = \begin{pmatrix} e^+ \\ \overline{\nu} \end{pmatrix} \tag{1}$$

The representation of the right-handed particle creation operators in the standard model can then be written

$$\begin{aligned} &u^\dagger : (3,1)_{2/3}, \quad d^\dagger : (3,1)_{-1/3}, \quad e^\dagger : (1,1)_{-1}, \\ &\overline{\psi}^\dagger : (\overline{3},2)_{-1/6}, \quad \overline{l}^\dagger : (1,2)_{1/2} \end{aligned} \tag{2}$$

where the numbers in parenthesis represent the SU(3) and SU(2) dimensions respectively. The subscript is the hypercharge of the U(1) generator S. These are combined into the full SU(3)×SU(2)×U(1) representation:

$$(3,1)_{2/3} \oplus (3,1)_{-1/3} \oplus (1,1)_{-1} \oplus (\overline{3},2)_{-1/6} \oplus (1,2)_{1/2} \tag{3}$$

For the left-handed particle creation operators the representation is simply the complex conjugate.

$$(\overline{3},1)_{-2/3} \oplus (\overline{3},1)_{1/3} \oplus (1,1)_{1} \oplus (3,2)_{1/6} \oplus (1,2)_{-1/2} \tag{4}$$

where $2 = \overline{2}$ has been used.

3 SU(5) unification

Obviously SU(5) has five dimensional fundamental representations. Since this 5 is complex, there is an additional fundamental representation $\overline{5}$. We now want to find an SU(3)×SU(2)×U(1) subgroup of SU(5) such that the 5 becomes a subgroup of the creation operators in Eq. 3. There are two possible five-dimensional subsets of Eq. 3:

$$(3,1)_{-1/3} \oplus (1,2)_{1/2}, \tag{5}$$

$$(3,1)_{2/3} \oplus (1,2)_{1/2} \tag{6}$$

However, for the second of these it is not possible to embed U(1) in SU(5) in such a way that S is traceless. On the other hand, we can easily embed SU(3)×SU(2)×U(1) in SU(5) to obtain the first one. The SU(3) generators T_a act on the first three indeces, the SU(2) generators R_a act on the last two, and the U(1) generator S commutes with the other generators.

$$\begin{pmatrix} T_a & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & R_a \end{pmatrix}, \qquad \begin{pmatrix} -I/3 & 0 \\ 0 & I/2 \end{pmatrix} \tag{7}$$

Since T_a and R_a are traceless all of the above matrices are traceless, as required. It is consequently possible to put the creation operators of the right-handed particles into an SU(5) multiplet:

$$\begin{pmatrix} d_r \\ d_g \\ d_b \\ \overline{e} \\ \overline{\nu} \end{pmatrix} \tag{8}$$

To obtain the remaining ten-dimensional part of Eq. 3, another representation is needed. SU(5) has two ten-dimensional representations, 10 and $\overline{10}$. Since $\overline{10} = \overline{5} \otimes_{\mathrm{AS}} \overline{5}$, the transformation of $\overline{10}$ under SU(3)×SU(2)×U(1) is given by taking the antisymmetric product of Eq. 5 with itself:

$$\begin{aligned} \left[(\overline{3},1)_{1/3} \oplus (1,2)_{-1/2}\right] \otimes_{\mathrm{AS}} \left[(\overline{3},1)_{1/3} \oplus (1,2)_{-1/2}\right] &= \\ = \left[(3,1)_{2/3} \oplus (1,1)_{-1} \oplus (\overline{3},2)_{-1/6}\right] \end{aligned} \tag{9}$$

since $3 \otimes 3 = \overline{3} \oplus 6$ and $2 \otimes 2 = 1 \oplus 3$ where the first term in each case is the antisymmetric part. Consequently $\overline{10}$ is the representation needed to get the rest of Eq. 3 and the rest of the creation operators can be put into a matrix:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_b & -u_g & -\overline{u}_r & -\overline{d}_r \\ -u_g & 0 & u_r & -\overline{u}_g & -\overline{d}_g \\ u_g & -u_r & 0 & -\overline{u}_b & -\overline{d}_b \\ \overline{u}_r & \overline{u}_g & \overline{u}_b & 0 & e \\ \overline{d}_r & \overline{d}_g & \overline{d}_b & e & 0 \end{pmatrix} \tag{10}$$

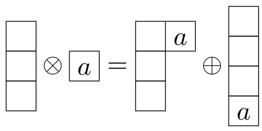
The right-handed particle creation operators have thus been embedded into a $5 \oplus \overline{10}$ representation. The left-handed can similarly be embedded into a $\overline{5} \oplus 10$ representation.

4 Breaking of SU(5)

The breaking of SU(5) into SU(3)×SU(2)×U(1) can be done in a similar way as the breaking of SU(3) into SU(2)×U(1). The S generator in the adjoint representation of SU(5) commutes with SU(3)×SU(2)×U(1). The 24 adjoint representation with vacuum values in the S direction will consequently break SU(5) into SU(3)×SU(2)×U(1). Under SU(3)×SU(2)×U(1) the adjoint 24 representation transforms as:

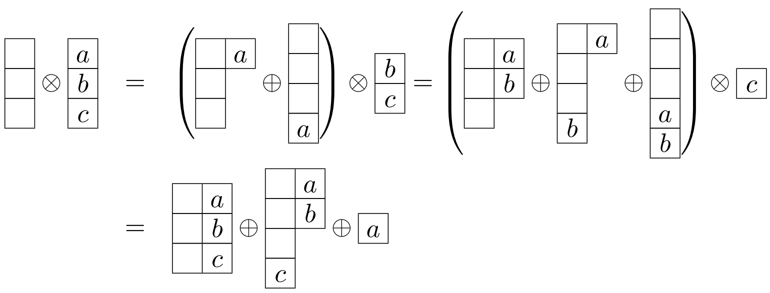
$$24 \rightarrow (8,1)_0 \oplus (1,3)_0 \oplus (1,1)_0 \oplus (3,2)_{-5/6} \oplus (\overline{3},2)_{5/6} \tag{11}$$

The first three terms can be identified with the standard model gauge bosons and the last two correspond to additional gauge bosons which are both SU(2) doublets and SU(3) triplets.
In order for a fermion to get mass in the SU(5) theory, the product of the representation containing the fermion and the representation containing the anti-fermion must include a component which transforms like the SU(3)×SU(2)×U(1) Higgs field, i.e. $(1,2)_{1/2}$, or its complex conjugate.
For right-handed d quarks and positrons, the particles and anti-particle appear in the two different representations 5 and $\overline{10}$. The product of these representations can be determined using Young tableaux.



(12)

By studying the dimensions of the Young tableaux above we get $5 \otimes \overline{10} = \overline{5} \oplus \overline{45}$. From Eq. 5 it is clear that 5 contains $(1,2)_{1/2}$ and it can also be shown that it is contained in 45. Consequently, both of these representations can be used to give mass to d and e.
For right-handed u quarks, both the particles and anti-particles are in $\overline{10}$. Again using Young tableaux, we get


(13)

By once again studying the dimensions of the Young tableaux, it can be shown that $10 \otimes 10 = 5 \oplus 45 \oplus 50$. As before 5 and 45 can be used to give mass to the u quarks. On the other hand 50 cannot, since it does not include $(1,2)_{1/2}$.

5 Physical consequences of the SU(5) theory

In the SU(5) theory the charge of the d quark can be deduced from the fact that there are three colour states. Since the charge operator $Q = R_3 + S$ it must be traceless. Looking at the multiplet for the 5 representation:

$$Q(\overline{\nu}) + Q(e^+) + 3Q(d) = 0 \qquad \Rightarrow \qquad Q(d) = -\frac{1}{3}Q(e^+) \tag{14}$$

Another success of the SU(5) theory is the reasonably accurate prediction of the Weinberg angle, giving $\sin^2\theta_W \approx 0.21$ at the electroweak scale.
The three coupling constants of the standard model are energy dependent. In the SU(5) it is possible calculate that they unite at an energy of approximately 10^{15} GeV. However, to get an exact unification of the coupling constants in a single point, a supersymmetric SU(5) theory is needed.
SU(5) has $5^2 - 1 = 24$ generators, which means that there are 24 gauge bosons instead of the usual 12. The additional gauge bosons are called X and Y and violate baryon and lepton numbers. As a consequence the proton can decay to a positron and a neutral pion. No such decay has ever been observed which is in contradiction to the proton lifetime estimated in the SU(5) theory. In a supersymmetric SU(5) theory the proton lifetime is longer and consistent with experimental data.

6 References

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