Mass relations

Project in the course Group Theory

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SU(3) flavour

Assumptions: 1) Strong force is flavour independent
2) u,d,s quarks have the same mass

Hamiltonian of the strong force is invariant under SU(3) transformations of the quarks

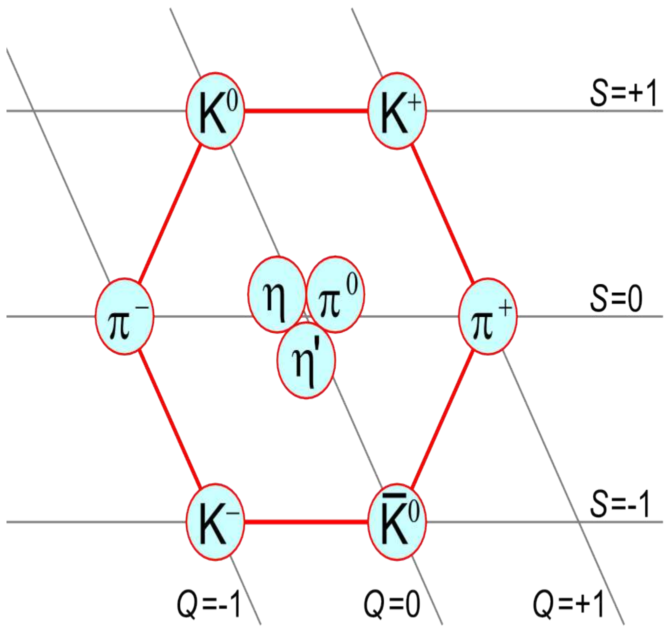
Isospin I and hypercharge are conserved and form a SU(2) $\times$ U(1) subgroup

$[I_a, I_b] = i\varepsilon_{abc} I_c$

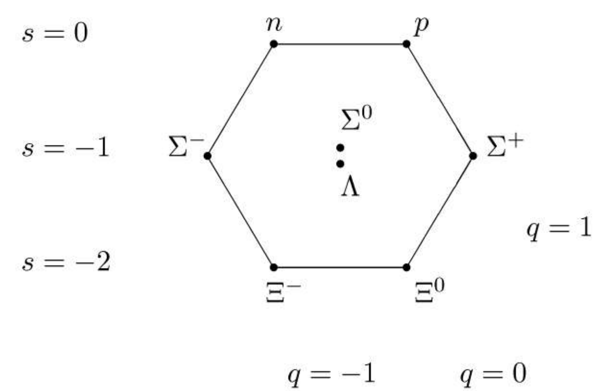
$[I_a, Y] = 0$

Hadrons

Mesons $q\bar{q}$



Baryons qqq



$$3 \times 3^* = 1 + 8$$

$$3 \times 3 \times 3 = 10 + 8 + 8 + 1$$

Mass splitting

$m_u = m_d < m_s$ (mass of u and d quarks equal and smaller than s)

$$\left(\langle q_i|\boldsymbol{H}_{st}|q_j\rangle\right) = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$
$$= \frac{2m_u + m_s}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{m_u - m_s}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$
$$= \frac{2m_u + m_s}{3}1 + \frac{m_u - m_s}{\sqrt{3}}\lambda_8$$

If the strength of the non invariant part is smaller than the invariant, pertubation theory can be used.

$$m_B = m^{(0)} + \langle \boldsymbol{B}^{(0)}|\boldsymbol{H}_8|\boldsymbol{B}^{(0)}\rangle$$

The generalised Wigner-Eckart theorem tells us to find all matrices that transform like the eighth component of an octet

$$\left(\langle \boldsymbol{B}'^{(0)}|\boldsymbol{H}_8|\boldsymbol{B}^{(0)}\rangle\right)$$

The generators of the representation of the SU(3) transform like an octet

$$\boldsymbol{F}_a (a = 1, ..., 8)$$

Another set can be formed with the symmetric symbol d_{abc}

$$\boldsymbol{D}_a = d_{abc}\boldsymbol{F}_b\boldsymbol{F}_c$$

Which leads to:

$$\langle \boldsymbol{B}'^{(0)} | \boldsymbol{H}_8 | \boldsymbol{B}^{(0)} \rangle = \langle \boldsymbol{B}'^{(0)} | \delta m_1' \boldsymbol{F}_8 + \delta m_2' d_{8ab} \boldsymbol{F}_a \boldsymbol{F}_b | \boldsymbol{B}^{(0)} \rangle$$

Looking closer on the second term

$$\begin{aligned} d_{8ab}\boldsymbol{F}_a\boldsymbol{F}_b &= \frac{1}{\sqrt{3}}(\boldsymbol{F}_1^2 + \boldsymbol{F}_2^2 + \boldsymbol{F}_3^2) - \frac{1}{2\sqrt{3}}(\boldsymbol{F}_4^2 + \boldsymbol{F}_5^2 + \boldsymbol{F}_6^2 + \boldsymbol{F}_7^2) - \frac{1}{\sqrt{3}}\boldsymbol{F}_8^2 \\ &= -\frac{1}{2\sqrt{3}}(\boldsymbol{F}_a\boldsymbol{F}_a) + \frac{3}{2\sqrt{3}}(\boldsymbol{F}_1^2 + \boldsymbol{F}_2^2 + \boldsymbol{F}_3^2) - \frac{1}{2\sqrt{3}}\boldsymbol{F}_8^2 \end{aligned}$$

Identifying the generators using the formula

$$\boldsymbol{F}_a|q_i\rangle = |q_j\rangle(\frac{\lambda_a}{2})_{ji}$$

Inserting gives:

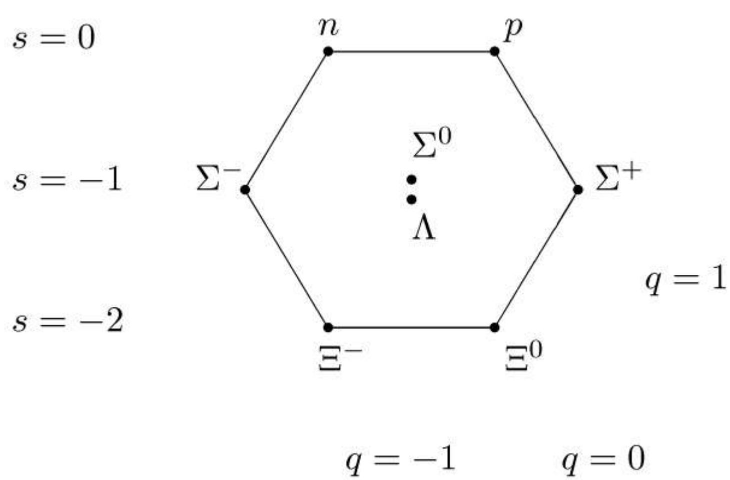
$$\langle \boldsymbol{B}'^{(0)}|\boldsymbol{H}_8|\boldsymbol{B}^{(0)}\rangle = \langle \boldsymbol{B}'^{(0)}|\delta m_1 \boldsymbol{Y} + \delta m_2(\vec{\boldsymbol{I}}^2 - \frac{1}{4}\boldsymbol{Y}^2 - \frac{1}{3}\boldsymbol{F}_a\boldsymbol{F}_a)|\boldsymbol{B}^{(0)}\rangle$$

Leading to:

$$m_b = m_0 + \delta m_1 Y + \delta m_2 (I(I+1) - \frac{1}{4}Y^2)$$

Which is the Gell-Mann-Okubo mass formula

The baryon octet

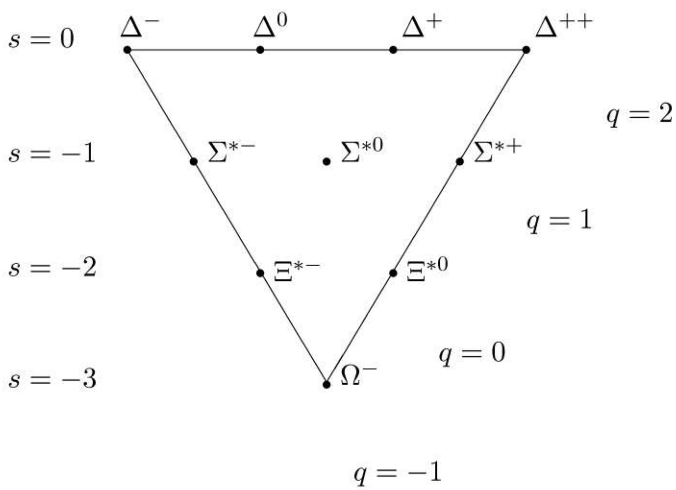


Baryon	Y	I	Mass (MeV)	Mass formula
p	1	$\frac{1}{2}$	938	$m_0 + \delta m_1 + \frac{1}{2}\delta m_2$
n	1	$\frac{1}{2}$	939	$m_0 + \delta m_1 + \frac{1}{2}\delta m_2$
Λ	0	0	1116	m_0
Σ^+	0	1	1189	$m_0 + 2\delta m_2$
Σ^0	0	1	1193	$m_0 + 2\delta m_2$
Σ^-	0	1	1197	$m_0 + 2\delta m_2$
Ξ^0	-1	$\frac{1}{2}$	1315	$m_0 - \delta m_1 + \frac{1}{2}\delta m_2$
Ξ^-	-1	$\frac{1}{2}$	1321	$m_0 - \delta m_1 + \frac{1}{2}\delta m_2$

$$\frac{m_N + m_\Xi}{2} = \frac{3m_\Lambda + m_\Sigma}{4}$$

1129 MeV 1135 MeV

The Baryon decuplet

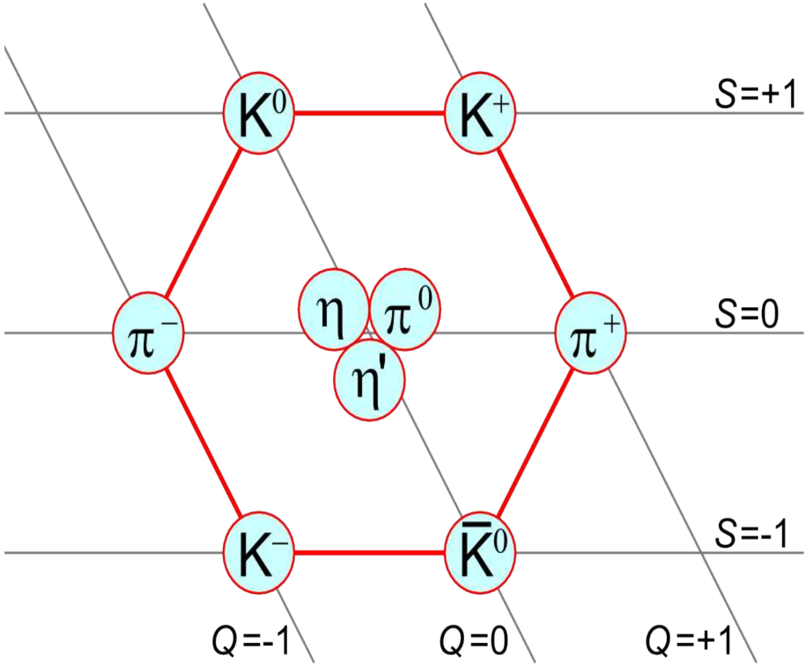


Baryon	Y	I	Mass (MeV)	Mass formula
Δ^{++}	1	$\frac{3}{2}$	1232	$m_0 + \delta m_1 + \frac{7}{2}\delta m_2$
Δ^{+}	1	$\frac{3}{2}$	1232	$m_0 + \delta m_1 + \frac{7}{2}\delta m_2$
Δ^{0}	1	$\frac{3}{2}$	1232	$m_0 + \delta m_1 + \frac{7}{2}\delta m_2$
Δ^{-}	1	$\frac{3}{2}$	1232	$m_0 + \delta m_1 + \frac{7}{2}\delta m_2$
Σ^{*+}	0	1	1383	$m_0 + 2\delta m_2$
Σ^{*0}	0	1	1384	$m_0 + 2\delta m_2$
Σ^{*-}	0	1	1387	$m_0 + 2\delta m_2$
Ξ^{*0}	-1	$\frac{1}{2}$	1532	$m_0 - \delta m_1 + \frac{1}{2}\delta m_2$
Ξ^{*-}	-1	$\frac{1}{2}$	1535	$m_0 - \delta m_1 + \frac{1}{2}\delta m_2$
Ω^{-}	-2	0	1672	$m_0 - 2\delta m_1 - \delta m_2$

Difference in mass between the different hypercharges about 150 MeV

Gell-Mann used this to predict the mass, Isospin and Hypercharge of *Ω*

The meson octet and singlet



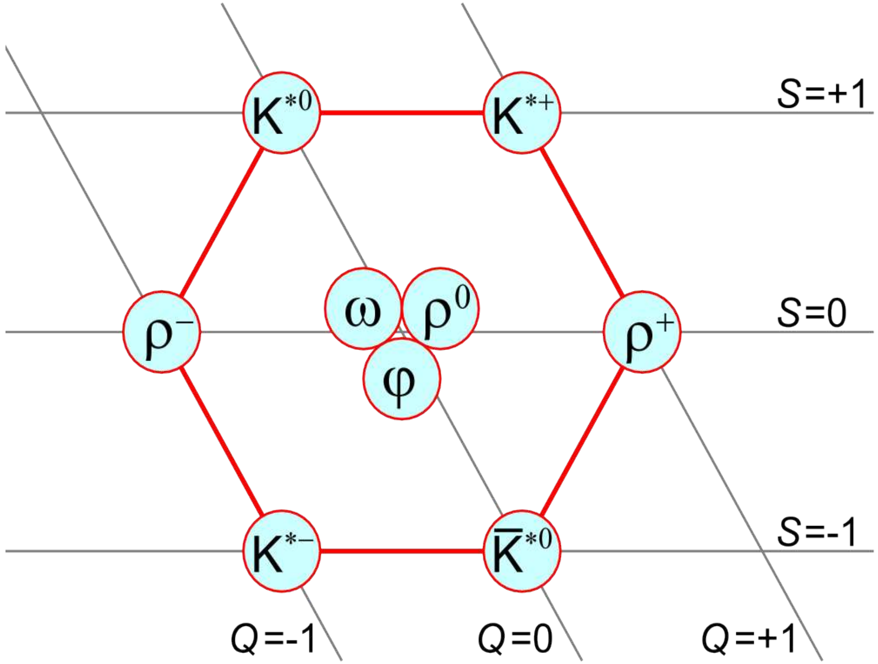
Meson	Y	I	Mass (MeV)	Mass formula
π^+	0	1	140	$m_8 + 2\delta m$
π^0	0	1	140	$m_8 + 2\delta m$
π^-	0	1	135	$m_8 + 2\delta m$
K^+	1	$\frac{1}{2}$	494	$m_8 + \frac{1}{2}\delta m$
K^0	1	$\frac{1}{2}$	498	$m_8 + \frac{1}{2}\delta m$
$\bar{K}^0$	-1	$\frac{1}{2}$	498	$m_8 + \frac{1}{2}\delta m$
K^-	-1	$\frac{1}{2}$	494	$m_8 + \frac{1}{2}\delta m$
η	0	0	548	–
η'	0	0	956	–

η and η' are a mix of the members of the octet and singlet

$$|\eta\rangle = \cos(\theta)|\eta_8\rangle + \sin(\theta)|\eta_1\rangle$$
$$|\eta'\rangle = -\sin(\theta)|\eta_8\rangle + \cos(\theta)|\eta_1\rangle$$

with mixing angle $\theta = 24^\circ$

Meson nonet with spin 1



Meson	Y	I	Mass (MeV)	Mass formula
ρ^+	0	1	776	$m_8 + 2\delta m$
ρ^0	0	1	776	$m_8 + 2\delta m$
ρ^-	0	1	776	$m_8 + 2\delta m$
K^{*+}	1	$\frac{1}{2}$	896	$m_8 + \frac{1}{2}\delta m$
K^{*0}	1	$\frac{1}{2}$	840	$m_8 + \frac{1}{2}\delta m$
$\bar{K}^{*0}$	-1	$\frac{1}{2}$	840	$m_8 + \frac{1}{2}\delta m$
K^{*-}	-1	$\frac{1}{2}$	896	$m_8 + \frac{1}{2}\delta m$
ω	0	0	783	–
ϕ'	0	0	1019	–

The mixing angle of the ω,ϕ is 53º

Conclusions

- Group theory can be used to derive the mass splititing within baryon octet and decuplet and meson nonet.
- The mixing angles of η,η' and ω,ϕ can be calculated
- Gell-Mann used this to predict the existence of the last member of the baryon decuplet