

# Group theory in physics

Symmetry of a physical problem leads to conserved quantities, or good quantum numbers. Group theory is the mathematics to deal with symmetry.

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## 1 Groups

### Definition of a group

A set  $\{a, b, c, \dots\}$  forms a group  $\mathcal{G}$  if the following conditions are all fulfilled

1. There exists an associative composition law among the elements:  $a \cdot b$ , where  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ . (Usually we will skip the composition sign  $\cdot$ , i.e.  $ab \equiv a \cdot b$ .)
2. For  $a, b \in \mathcal{G}$  there exist only one  $c = a \cdot b \in \mathcal{G}$ .
3. There exist an identity among the elements:  $e \in \mathcal{G}$ , for which  $e \cdot a = a$  for all  $a \in \mathcal{G}$ .
4. The inverse of each element also belong to the set:  
 $a \in \mathcal{G} \Rightarrow a^{-1} \in \mathcal{G}$  where  $a^{-1} \cdot a = e$ .

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If all its elements commute,  $ab = ba \quad \forall a, b \in \mathcal{G}$ , we have an *Abelian* group.

A *subgroup*  $\mathcal{H}$  of  $\mathcal{G}$  is a subset of its elements that also forms a group,  $\mathcal{H} \subset \mathcal{G}$ . There are always two trivial subgroups,  $\mathcal{H} =$

$\{e\}$  and  $\mathcal{H} = \mathcal{G}$ . Other subgroups are referred to as *proper* subgroups.

The *order* of a group is equal to the number of its elements,  $g = \text{ord } \mathcal{G}$ .

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In a mathematical sense a group is an abstract object which is defined through its *table of compositions* (or composition laws). It is referred to as the *group table*, *composition table*, or *multiplication table*.

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Example The group of the triangle or the ammonia molecule. There are threefold rotations by  $120^\circ$ ,  $c_3$ , and  $240^\circ$ ,  $c_3^2$ , respectively, counter-clockwise around the vertical axis going through the triangle's center of gravity. There are three mirroring in planes perpendicular to the triangle plane and through one of its corners and bisecting the triangle,  $\sigma_a$ ,  $\sigma_b$  and  $\sigma_c$ , respectively. And then we have the identity, not doing anything. These operations form a group  $C_{3v} = \{e, c_3, c_3^2, \sigma_a, \sigma_b, \sigma_c\}$ . (Verify that all inverses are included.)

When these operations are performed after each other (composition) we get the composition table. For instance:  $c_3\sigma_a = \sigma_c$ . This means that a mirroring in the  $a$ -plane followed by  $120^\circ$  rotation (note that the right operator acts first) is equivalent with a mirroring in the  $c$ -plane. Going through all possible composition  $ab = c$  generates a table labeled by the row  $a$  and column  $b$ :

$C_{3v}$	$e$	$c_3$	$c_3^2$	$\sigma_a$	$\sigma_b$	$\sigma_c$
$e$	$e$	$c_3$	$c_3^2$	$\sigma_a$	$\sigma_b$	$\sigma_c$
$c_3$	$c_3$	$c_3^2$	$e$	$\sigma_c$	$\sigma_a$	$\sigma_b$
$c_3^2$	$c_3^2$	$e$	$c_3$	$\sigma_b$	$\sigma_c$	$\sigma_a$
$\sigma_a$	$\sigma_a$	$\sigma_b$	$\sigma_c$	$e$	$c_3$	$c_3^2$
$\sigma_b$	$\sigma_b$	$\sigma_c$	$\sigma_a$	$c_3^2$	$e$	$c_3$
$\sigma_c$	$\sigma_c$	$\sigma_a$	$\sigma_b$	$c_3$	$c_3^2$	$e$

Verify that the group is non-Abelian and that it has two proper subgroups:  $C_3 = \{e, c_3, c_3^2\}$  and  $C_s = \{e, \sigma_a\}$  (or equivalently, with  $\sigma_b$  or  $\sigma_c$ ).

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### Other examples of groups

- Integers under addition:  $n + m = k$ ,  $n, m, k \in \mathbb{Z}$  form an Abelian group. Verify that this satisfies the criteria.
- Rotations in 2D (cylindrical symmetry) form an Abelian continuous group called  $SO(2)$ . The group elements are rotations  $\mathcal{R}(\theta)$  with the angle  $\theta \in [0, 2\pi)$ .
- Permutations of three particles form a group  $\mathcal{P}_3$ . It consists of six elements: identity – (1), three exchanges – (12) ( $1 \rightarrow 2 \rightarrow 1$ ), (13), and (23), and two cyclic permutations – (123) and (132). (The last ones should read  $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$  and  $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$ , respectively). Verify that this group has the same composition table as  $C_{3v}$ .

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The *generators* of a group is a small subset of elements (often the smallest) that together by successive compositions generates the full group. We can always define a group through its generators.

Ex. Another way of defining  $C_{3v}$  is through:

$$C_{3v} : p^3 = e, q^2 = e, (qp)^2 = e.$$

Where  $p$  is either  $c_{3v}$  or  $c_{3v}^2$  and  $q$  is one of the mirroring  $\sigma_i, i \in \{a, b, c\}$ . Verify that these two generators bring out the full group.

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A *cyclic* group is defined through one generator,  $C_n : p^n = e$ . Verify that a cyclic group always is Abelian.

The *order* of a generator is defined through the smallest power which generates the identity, i.e.  $p^n = e \Rightarrow n = \text{ord } p$ . For a cyclic group  $\text{ord } C_n = \text{ord } p = n$ .

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A mapping from one group to another group is *homomorphic* if it keeps the composition table. Thus for a homomorphic mapping  $f : \mathcal{G} \rightarrow \mathcal{H}$ ,

$$ab = c \Rightarrow f(a)f(b) = f(c)$$

with  $a, b, c \in \mathcal{G}$ . Then we say that  $\mathcal{H}$  is homomorphic to  $\mathcal{G}$  or  $\mathcal{H} \simeq \mathcal{G}$ . In general a homomorphic mapping can be many-to-one. If it is a one-to-one mapping it is called *isomorphic*, then  $\mathcal{H} \cong \mathcal{G}$ .

Ex.  $C_{3v} \cong \mathcal{P}_3$ .

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There are two important ways to divide a group into disjoint sets. Either through *cosets* or through *conjugate classes*.

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The *left coset* of subgroup  $\mathcal{H} \subset \mathcal{G}$ , divide  $\mathcal{G}$  into disjoint subsets through  $\mathcal{G} = \sum_i^g a_i \mathcal{H}$ . Note that some  $a_i \mathcal{H}$  generate same subset, for instance  $b\mathcal{H} = \mathcal{H}$  when  $b \in \mathcal{H}$ . The subgroup  $\mathcal{H}$  itself form one coset and it is possible to show that all cosets have the same dimension,  $h = \text{ord } \mathcal{H}$ . Then the number of cosets are  $n = g/h$ .

The *right coset* is similarly defined through  $\mathcal{G} = \sum_i^g \mathcal{H} a_i$ .

If the right and left cosets are the same,  $\mathcal{H}$  is a *normal* or *invariant* subgroup.

Since the identity is only in the coset of the subgroup, no other coset forms a group.

Ex.  $C_{3v}$  can be divided into left cosets with respect to its subgroup  $C_3$ :

$$C_{3v} = eC_3 \cup \sigma_a C_3 = \{e, c_3, c_3^2\} \cup \{\sigma_a, \sigma_b, \sigma_c\}$$

Verify that  $C_3$  is a normal subgroup of  $C_{3v}$ .

Ex.  $C_{3v}$  can be divided into left cosets with respect to its subgroup  $C_s = \{e, \sigma_a\}$ :

$$C_{3v} = eC_s \cup c_3 C_s \cup c_3^2 C_s = \{e, \sigma_a\} \cup \{c_3, \sigma_c\} \cup \{c_3^2, \sigma_b\}$$

Is  $C_s$  a normal subgroup?

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A group is called *simple* if it has no normal proper subgroup and it is called *semi-simple* if it has no Abelian normal proper subgroup.

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The *conjugate*,  $b$ , to  $a$  with respect to  $t$  is defined through;  $b = tat^{-1}$ . When  $t$  runs through all elements of  $\mathcal{G}$  the set  $\{tat^{-1}; \forall t \in \mathcal{G}\}$ , forms the *conjugate class* of  $a$ , or in short the *class* of  $a$ .

The conjugation leads to a division of the group elements into a set of disjoint classes.

It is possible to show that the number of elements,  $r_a$ , in a class,  $K_a$ , is a divisor of  $g = \text{ord } \mathcal{G}$  and that  $g = \sum_a r_a$ .

Ex.  $C_{3v}$  has three classes:  $K_1 = \{e\}$ ,  $K_2 = \{c_3, c_3^2\}$ , and  $K_3 = \{\sigma_i; i = a, b, c\}$ . Their sizes  $r_1 = 1$ ,  $r_2 = 2$  and  $r_3 = 3$  are all divisors of  $g = 6$  and their sum is 6. That the number of classes is a divisor too is just a coincidence.

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Often we are interested in combinations of groups. For instance, we can define an *outer direct group* as  $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ , where both  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are normal subgroups of  $\mathcal{G}$ . That is, we have an outer direct group if

1.  $\forall a_i \in \mathcal{G}_1$  and  $\forall b_j \in \mathcal{G}_2 \Rightarrow a_i b_j = b_j a_i \in \mathcal{G}$ .
2.  $\mathcal{G}_1 \cap \mathcal{G}_2 = \{e\}$ .
3. then  $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 = \mathcal{G}_2 \times \mathcal{G}_1$ .

These are usually of use when combining different kind of symmetries, which can act in different spaces, as spin and real space.

When we have several particles encountering the same symmetry, we deal with *inner direct products*. Then we have that  $\mathcal{G} = \mathcal{G}_1 \otimes \mathcal{G}_2$ , where  $\mathcal{G} \cong \mathcal{G}_1 \cong \mathcal{G}_2$ . That is  $a_i a_i \in \mathcal{G}$  if  $a_i \in \mathcal{G}_1 = \mathcal{G}_2$ .

Finally, of interest is also a *semi-direct* product,  $\mathcal{G} = \mathcal{G}_1 \wedge \mathcal{G}_2$ . For this group product we require,

1.  $\mathcal{G}_1$  is normal group to  $\mathcal{G}$ .
2.  $\mathcal{G}_1 \cap \mathcal{G}_2 = \{e\}$ .
3.  $a_i b_j \in \mathcal{G}$  if  $a_i \in \mathcal{G}_1$  and  $b_j \in \mathcal{G}_2$ .

Ex. It is possible to define the  $C_{3v}$  group as  $C_3 \wedge C_s$ .