Supersymmetry.

An introductory text of the Coleman-Mandula theorem and the supersymmetric algebra and harmonic oscillator.

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Introduction

The purpose of supersymmetry, among other things, is to provide a symmetry group that relates the symmetries of space-time and translations, internal symmetries and also other symmetries that relates fermions with bosons. Space-time and translation symmetries are given by the Poincaré group and the internal symmetries. The first questions we come to ask is how these groups can be related in a way that provides physics as we know it and how does the generators of the different subgroups commute. To start of we look to the Coleman-Mandula theorem that states how we can relate the Poincaré group with other symmetry groups. From there on we proceed in studying the algebra of the supersymmetry group. Finally we will consider the harmonic oscillator in relation to supersymmetry.

Coleman-Mandula theorem[1]

The Coleman-Mandula theorem is a *no-go theorem*, which is a type of theorem can be summarized in *"we cannot do what we want"*. In this case, we would want to connect the Poincaré group and other groups in complex and/or attractive ways and obtain physics as we can measure it and maybe new theoretical findings. But the Coleman-Mandula theorem is true to its classification as a no-go theorem and we cannot connect the Poincaré group with other groups in a non-trivial way. In every non-trivial connection we will end up with trivial physics.
To get something useful out of this we need to first state what kind of physics we want to obtain from this group, preferably with as few statements as possible, and then see to all the possible connections. When Coleman and Mandula proved the theorem they used the following four criteria[2]

1. A group $\mathcal{G}$ has a subgroup that locally is isomorphic to the Poincaré group $\mathcal{P}$.
2. For any mass M we have a finite number of particles with masses less than M, and all particles correspond to positive-energy representations of the Poincaré group.
3. The scattering amplitudes of elastic scattering should be analytical functions of center of mass energy and momentum transfer, which we physically expect.
4. We demand non-trivial scattering, i.e. we want scattering to have scattering angles other than $0°$ and $180°$.

These demands is sufficient to know that the resulting group $\mathcal{G}$ is locally isomorphic to the direct product of the Poincaré group $\mathcal{P}$ and a Lorentz-invariant compact group $\mathcal{T}$,[3] and that this is the only way to connect them. The theorem also implies that we can have additional $U(1)$ groups.[4]

[1] S. Coleman, J. Mandula, Phys. Rev. **159**, 5 (1967)
[2] There is actually a fifth criteria that we have to demand on $\mathcal{G}$, this criteria is however greatly topologically related, and for this paper we do not need to consider that.
[3] J. Wess, J. Bagger, *Supersymmetry and Supergravity*, p. 4 (1983)
[4] P. West, *Introduction to supersymmetry and supergravity*, p. 5 (1990)

The first point is just what we want, i.e. that one part of the group is the Poincaré group. The second, third and fourth is of physical origin. The properties that are demanded is what we know from experiments.

The supersymmetry algebra

From the Poincaré group $\mathcal{P}$ we have the Lorentz (L_{ab}) and momentum (P_a) generators with the commutation relations[5]

$$\begin{aligned}
[P_a, P_b] &= 0, \\
[P_a, L_{bc}] &= (\eta_{ab}P_c - \eta_{ac}P_b), \\
[L_{ab}, L_{cd}] &= -(\eta_{ac}L_{bd} + \eta_{bd}L_{ac} - \eta_{ad}L_{bc} - \eta_{bc}L_{ad}); \quad a,b,c,d \in \{0,1,2,3\}.
\end{aligned}$$

Where the Lorentz generators are divided into proper- (R_i) and pseudo- (K_i) rotation generators (K_i are also known as boosts)

$$\begin{aligned}
[R_i, R_j] &= i\varepsilon_{ijk}R_k, \\
[K_i, K_j] &= -i\varepsilon_{ijk}R_k, \\
[R_i, K_j] &= i\varepsilon_{ijk}K_k,
\end{aligned}
\qquad
\begin{aligned}
[P_0, R_i] &= 0, \\
[P_i, R_j] &= i\varepsilon_{ijk}P_k, \\
[P_0, K_i] &= iP_i, \\
[P_i, K_j] &= iP_0\delta_{ij},
\end{aligned}$$

related to the Lorentz generators through

$$\begin{aligned}
R_i &= \frac{1}{2}\varepsilon_{ijk}L_{jk}, \\
K_i &= L_{0i}; \qquad i,j,k \in \{1,2,3\}
\end{aligned}$$

As stated in the Coleman-Mandula theorem if we introduce a Lorentz-invariant compact group $\mathcal{T}$ with internal symmetries which we want to connect to the Poincaré group $\mathcal{P}$ the relation must be a direct product. This means that the generators P_a, L_{ab} and T_s of the two subgroups $\mathcal{P}$ and $\mathcal{T}$, respectively, of the supersymmetry group $\mathcal{G}$ must commute[6]

$$\begin{aligned}
[P_a, T_s] &= 0, \\
[L_{ab}, T_s] &= 0.
\end{aligned}$$

The important step to take here is to include an other set of generators Q^i_α ($i \in \{1, \ldots, N\}$, $\alpha \in \{1,2\}$), which we will later see that are the ones that relates bosons and fermions together. For these generators to preserve the antisymmetric properties that the fermions possess, we need to consider the anti-commutator

$$\{Q^i_\alpha, Q^j_\beta\} = Q^i_\alpha Q^j_\beta + Q^j_\beta Q^i_\alpha.$$

We are allowed to use these generators by demanding a $\mathbb{Z}_2$ gradation (we can divide the generators into two groups, even (zero degree) and odd (one degree)) on our group.[7] A $\mathbb{Z}_2$ gradation provides us, by definition, with some commutator relations

$$\begin{aligned}
[\text{even}, \text{even}] &= \text{even}, \\
\{\text{odd}, \text{odd}\} &= \text{even}, \\
[\text{even}, \text{odd}] &= \text{odd}. \qquad (1)
\end{aligned}$$

Where the odd generators are the Q^i_α's and the even generators are the others, i.e. L_{ab}, P_a and T_s. To get the complete supersymmetric algebra we need th study the odd-odd and even-odd

[5] H.F. Jones, *Groups, Representations and Physics*, p. 207-211 (1998)
[6] P. West, *Introduction to supersymmetry and supergravity*, p. 7 (1990)
[7] Y. A. Golfand, E. S. Likhtman, JETP Lett. **13**, 323 (1971)

commutators. Since there is only one set of odd generators we conclude that

$$\begin{aligned}
\left[Q^i_\alpha, L_{ab}\right] &= (a_{ab})^\beta_\alpha Q^i_\beta, \\
\left[Q^i_\alpha, P_a\right] &= (b_a)^\beta_\alpha Q^i_\beta, \\
\left[Q^i_\alpha, T_s\right] &= (c_s)^{\beta i}_{\alpha j} Q^j_\beta. \qquad (2)
\end{aligned}$$

The coefficient tensors is determined by the generalized Jacobi identities, provided to us from the $\mathbb{Z}_2$ gradation of the algebra

$$\begin{aligned}
[[\text{even}, \text{even}], \text{even}] + [[\text{even}, \text{even}], \text{even}] + [[\text{even}, \text{even}], \text{even}] = 0 \\
[[\text{even}, \text{even}], \text{odd}] + [[\text{odd}, \text{even}], \text{even}] + [[\text{even}, \text{odd}], \text{even}] = 0 \\
\{[\text{even}, \text{odd}], \text{odd}\} + \{[\text{even}, \text{odd}], \text{odd}\} + [\{\text{odd}, \text{odd}\}, \text{even}] = 0 \\
\{\{\text{odd}, \text{odd}\}, \text{odd}\} + \{\{\text{odd}, \text{odd}\}, \text{odd}\} + \{\{\text{odd}, \text{odd}\}, \text{odd}\} = 0
\end{aligned}$$

for three different generators and the first two with cyclic permutation and the second two with anti-cyclic permutation. We get the relations (2) on the form

$$\begin{aligned}
\left[Q^i_\alpha, L_{ab}\right] &= \frac{1}{2}(\sigma_{ab})^\beta_\alpha Q^i_\beta \\
\left[Q^i_\alpha, P_a\right] &= 0 \\
\left[Q^i_\alpha, T_s\right] &= (l_s)^i_j Q^j_\alpha + (t_s)^i_j (i\gamma_5)^\beta_\alpha Q^j_\beta
\end{aligned}$$

using a nomenclature for the tensors that is used in relevant literature. There is now one relation left to derive, the anti-commutator of the odd generators. From (1) we conclude that we should obtain a combination of all even generators that are symmetric under the change of indexes $\alpha \leftrightarrow \beta$ and $i \leftrightarrow j$, since the anti-commutator is symmetric in those indexes. We have

$$\{Q^i_\alpha, Q^j_\beta\} = r(\gamma^a C)_{\alpha\beta} P_a \delta^{ij} + s(\sigma^{ab} C)_{\alpha\beta} J_{ab} \delta^{ij} \qquad (3)$$

where we see that the indexes i and j looses it's importance due to the Kronecker delta ($N = 1$), and one of the Jacobi identities simplifies this further

$$\{Q_\alpha, Q_\beta\} = 2(\gamma^a C)_{\alpha\beta} P_a. \qquad (4)$$

Now we have found the supersymmetry algebra, that is we have the following commutation relations

$$\begin{aligned}
\left[Q_\alpha, L_{ab}\right] &= \frac{1}{2}(\sigma_{ab})^\beta_\alpha Q_\beta \\
\left[Q^i_\alpha, P_a\right] &= 0 \\
\left[Q_\alpha, T\right] &= i(\gamma_5)^\beta_\alpha Q_\beta \\
\{Q_\alpha, Q_\beta\} &= 2(\gamma^a C)_{\alpha\beta} P_a \qquad (5)
\end{aligned}$$

that characterizes the supersymmetry algebra. The commutation relation $[Q_\alpha, T]$ has here a new form due to the removal of indexes i and j.

However, the algebra in (5) is not the most general algebra for a supersymmetry. If we recall the Coleman-Mandula theorem we see that we can have additional $U(1)$ subgroups, so the most general form of (3) is actually

$$\{Q^i_\alpha, Q^j_\beta\} = r(\gamma^a C)_{\alpha\beta} P_a \delta^{ij} + s(\sigma^{ab} C)_{\alpha\beta} J_{ab} \delta^{ij} + C_{\alpha\beta} U^{ij} + (\gamma_5 C)_{\alpha\beta} V^{ij}.$$

One inherits a more complex algebra from this, called *the extended supersymmetry algebra*, one obtains

$$\begin{aligned}
\left[Q_A, L_{ab}\right] &= \frac{1}{2}(\sigma_{ab})^B_A Q^i_B \\
\left[Q^i_A, P_a\right] &= 0 \\
\left[Q^i_A, T_s\right] &= (l_r + it_r)^i_j Q^j_A \\
\left\{Q^i_A, \bar{Q}_{\dot{B}j}\right\} &= -2i(\sigma^a)_{A\dot{B}} \delta^i_j P_a \\
\left\{Q^i_A, \bar{Q}^j_B\right\} &= \varepsilon_{AB}(U^{ij} + iV^{ij}); \qquad A, B, \dot{B} = 1, 2
\end{aligned}$$

using the notion of a two component spinor. However we will not continue to discuss the extended supersymmetry algebra any further.

Supersymmetric harmonic oscillator

The usual quantum mechanical harmonic oscillator is characterized by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 x^2}{2}.$$

This can be solved in several ways, one is to use the creation $b^\dagger$ and annihilation b operators[8] defined by

$$b^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i\hat{p}}{m\omega}\right), \qquad b = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i\hat{p}}{m\omega}\right).$$

Which gives a the Hamiltonian as

$$\hat{H}_B = \hbar\omega\left(b^\dagger b + \frac{1}{2}\right).$$

This provides us with the familiar energy eigenvalues

$$\hat{H}_B |\, n\,\rangle = E_n^B |\, n\,\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|\, n\,\rangle\,.$$

The quantum number n, that labels each state is interpreted as a state occupied by n quanta[9], and the ground state energy is $\frac{1}{2}\hbar\omega$ because of the uncertainty principle.[10] These creation and annihilation operators raises and lowers the state, respectively, i.e.

$$b^\dagger |\, n\,\rangle = \sqrt{n+1}|\, n+1\,\rangle\,, \quad b|\, n\,\rangle = \sqrt{n}|\, n-1\,\rangle\,.$$

This provides us with the commutation relation

$$\left[b, b^\dagger\right] = 1.$$

However, we realize that this cannot be the case for fermions. In the case of fermions only one quanta can occupy the same state simultaneous[11], so we should only have the quantum numbers $k = 0, 1$. Now we introduce fermionic creation $f^\dagger$ and annihilation f operators, these gives a Hamiltonian

$$\hat{H}_F = \hbar\omega\left(f^\dagger f - \frac{1}{2}\right).$$

If we here impose nilpotence on these operators, i.e.

$$f^2 = 0 = {f^\dagger}^2$$

they obey the Pauli principle. The corresponding energy eigenvalues are

$$\hat{H}_F |\, k\,\rangle = E_n^F |\, k\,\rangle = \hbar\omega\left(k - \frac{1}{2}\right)|\, k\,\rangle$$

and we also see that they obey the following anti-commutation rule

$$\begin{aligned}\left\{f, f^\dagger\right\}|\, k\,\rangle &= f f^\dagger |\, k\,\rangle + f^\dagger f |\, k\,\rangle = \begin{cases} f|\, k+1\,\rangle + 0 = |\, k\,\rangle\,; & k=0 \\ 0 + f^\dagger |\, k-1\,\rangle = |\, k\,\rangle\,; & k=1 \end{cases} \\ &\Rightarrow \left\{f, f^\dagger\right\} = 1.\end{aligned}$$

$$\hbar\omega\left(b^\dagger b+\frac{1}{2}\right)|\, n\,\rangle=\hbar\omega\left(n+\frac{1}{2}\right)|\, n\,\rangle\,,\qquad \hbar\omega\left(f^\dagger f-\frac{1}{2}\right)|\, k\,\rangle=\hbar\omega\left(k-\frac{1}{2}\right)|\, k\,\rangle$$

Figure 1: The bosonic- and fermionic- harmonic oscillator energy eigenvalues in units of $\hbar\omega$.

In *Figure 1* we have the bosonic oscillator to the left and the fermionic oscillator to the right, with their corresponding Hamiltonians and eigenvalues. Now we want to create a connection between these two oscillators, in such a way that we can relate bosons to fermions and vice versa. The creation (annihilation) operator are interpreted as creating (annihilating) a new quanta and to relate bosons to fermions we should be considering the operators

$$Q=b^\dagger f,\quad Q^\dagger=f^\dagger b.$$

Where Q annihilates a fermion and creates a boson, and $Q^\dagger$ annihilates a boson and creates a fermion. If we study the anti-commutator of these two operators we notice

$$\begin{aligned}
\left\{Q^\dagger,Q\right\} &= Q^\dagger Q+QQ^\dagger=\left(Q^\dagger+Q\right)^2=\left(b^\dagger f+f^\dagger b\right)^2\\
&\quad\text{where}\\
\left(b^\dagger f+f^\dagger b\right)^2|\, 0,n\,\rangle &= \left(b^\dagger f+f^\dagger b\right)\left(b^\dagger f|\, 0,n\,\rangle+f^\dagger b|\, 0,n\,\rangle\right)=\sqrt{n}\left(b^\dagger f+f^\dagger b\right)|\, 1,n-1\,\rangle\\
&= \sqrt{n}b^\dagger f|\, 1,n-1\,\rangle=n|\, 0,n\,\rangle=\left(b^\dagger b+f^\dagger f\right)|\, 0,n\,\rangle\\
&\quad\text{and}\\
\left(b^\dagger f+f^\dagger b\right)^2|\, 1,n\,\rangle &= \left(b^\dagger f+f^\dagger b\right)\left(b^\dagger f|\, 1,n\,\rangle+f^\dagger b|\, 1,0\,\rangle\right)=\sqrt{n+1}\left(b^\dagger f+f^\dagger b\right)|\, 0,n+1\,\rangle\\
&= \sqrt{n+1}f^\dagger b|\, 0,n+1\,\rangle=(n+1)|\, 1,n\,\rangle=\left(b^\dagger b+f^\dagger f\right)|\, 1,n\,\rangle\\
\Rightarrow &\quad \left(b^\dagger f+f^\dagger b\right)^2=\left(b^\dagger b+f^\dagger f\right)=\frac{1}{\hbar\omega}\left(\hat{H}_B+\hat{H}_F\right)
\end{aligned}$$

That is we can now form a Hamiltonian

$$\hat{H}=\hat{H}_B+\hat{H}_F=\hbar\omega\left(b^\dagger b+f^\dagger f\right)=\hbar\omega\left\{Q^\dagger,Q\right\}\tag{6}$$

with the eigenvalue equation

$$\hat{H}|\, k,n\,\rangle=\hbar\omega(n+k)|\, k,n\,\rangle\,.$$

In *Figure 2* we find an illustration of the energy spectrum. We see that the energy states are

[8] J. J. Sakurai, *Modern Quantum Mechanics*, p. 90 (1994)
[9] E. Poppitz, *Dynamical supersymmetry breaking - why and how*, hep-ph/9710274v1 (1997)
[10] F. Mandl, *Statistical Physics*, p. 150-151 (1988)
[11] http://en.wikipedia.org/wiki/Pauli_exclusion_principle, *Pauli exclusion principle* (03-06-2007)

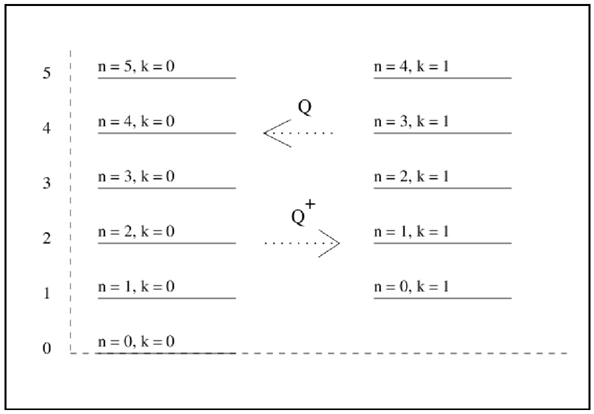

Figure 2: The supersymmetric harmonic oscillator.

degenerate, the state with n bosonic quanta and 0 fermionic quanta ($\mid 0, n\,\rangle$) have the same eigenvalue as the state with $n-1$ bosonic- and 1 fermionic- quanta ($\mid 1, n-1\,\rangle$). This is something which we will discuss later.

Discussion

In the section of the supersymmetry algebra we found the anti-commutation relation (4)

$$\{Q_\alpha, Q_\beta\} = 2(C\gamma_a)_{\alpha\beta}P^a$$

and in the section of the harmonic oscillator we found a similar anti-commutator to be proportional to the Hamiltonian of the system (6). As we know from our studies of special relativity we know that the zeroth component of the momentum vector P^a corresponds to the energy. And from our one dimensional example we see that we could preform a simple generalization

$$\begin{array}{ccc} \{Q^\dagger, Q\} & \longrightarrow & \{Q_\alpha, Q_\beta\} \\ E \propto P^0 & \longrightarrow & P^a \end{array}$$

which would provide us with the more exact relation, up to the coefficient tensors.

The degeneration found in the energy states of the supersymmetric oscillator is interpreted as a *super-multiplet*, that is we have a multiplet of one boson and one fermion that are related via the operators $Q^\dagger$ and Q. Here we see a hint to the minimal supersymmetric standard model (MSSM). The MSSM contains, in addition to the particles of the standard model, also the supersymmetric super-multiplet counterparts. For examples we have the photon (a boson) and the electron (a fermion), in their super-multiplets we the photino (the fermionic photon) and the selectron (the bosonic electron). There is a supersymmetric particle for each standard model particle.

References

1. S. Coleman, J. Mandula, Phys. Rev. **159**, 5 (1967).

2. J. Wess, J. Bagger, *Supersymmetry and Supergravity*, p. 4 (1983).

3. H.F. Jones, *Groups, Representations and Physics*, p. 207-211 (1998).

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9. http://en.wikipedia.org/wiki/Pauli_exclusion_principle, *Pauli exclusion principle* (03-06-2007).