Young tableaux

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Abstract

This paper gives a qualitative description how Young tableaux can be used to perform a Clebsch-Gordan decomposition of tensor products in SU(3) and how this can be generalized to SU(N).

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1 Young tableaux and tensors

While working directly with group representations in the case of finite groups, it has proven to be easier to work with the tensors spanning the corresponding vector spaces in the case of infinite groups. It can be shown that irreducible representations always correspond to a certain stage of symmetrization or antisymmetrization of the indices of such tensors, i.e. transforming a symmetric tensor results in a symmetric tensor – hence these tensors form an invariant subspace and the corresponding representation is irreducible. Take a tensor of rank 2, for example

$$\sigma_{ij} = \frac{1}{2}(\underbrace{\sigma_{ij} + \sigma_{ji}}_{\sigma_{(ij)}} + \underbrace{\sigma_{ij} - \sigma_{ij}}_{\sigma_{[ij]}})$$

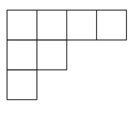
which can be decomposed into a symmetric ($\sigma_{(ij)}$) and an antisymmetric ($\sigma_{[ij]}$) part. These transform according to ${U_a}^i {U_b}^j \sigma_{ij}$. The transformed symmetric part is again symmetric and the transformed antisymmetric is again antisymmetric, as can be seen from

$$\begin{aligned} \sigma'_{(ba)} &= {U_b}^i {U_a}^j \sigma_{(ij)} = {U_a}^j {U_b}^i \sigma_{(ji)} = {U_a}^i {U_b}^j \sigma_{(ij)} = \sigma'_{(ab)} \\ \sigma'_{[ba]} &= {U_b}^i {U_a}^j \sigma_{[ij]} = -{U_a}^j {U_b}^i \sigma_{[ji]} = -{U_a}^i {U_b}^j \sigma_{[ij]} = -\sigma'_{[ab]} \end{aligned}$$

In the case of SU(2) (i.e. coupling of two 1/2 spins), there are only two components (possible index values). Hence in the antisymmetrized part the index values are fixed, rather than being variable. Therefore this corresponds to the familiar (antisymmetric) spin-0 singlett, while the symmetric part of the tensor corresponds to the (symmetric) spin-1 part.

Furthermore, in the case of SU(2) the representations corresponding to upper and lower indices are equivalent. For $N > 2$, they are not, however. In SU(3), lower indices are associated with particles (quarks), while upper indices are associated with anti-particles, since they transform according to the conjugate representation. It can be shown in general that in SU(N), $N-1$ antisymmetrized lower indices can be associated with 1 upper index.

For larger N, it gets increasingly difficult to construct the tensor products explicitly. However, Young tableaux provide a very useful graphical method. A young tableau can be represented by left-aligned boxes, where every line must not have more boxes than the preceding line. A Young tableau always represents a specific symmetrization/antisymmetrization of a tensor of rank n, where n is given by the number of boxes in the tableau. A valid tableau representing a tensor of rank 7 is:



To obtain the mentioned symmetries, one can think of the cells being initially filled by numbers ranging from $1 \leq i \leq n$ representing the i^{th} index from left to right and from top to bottom. To this tableau, the Young operator Y is applied to obtain the corresponding symmetry. This operator is given by

$$Y = QP$$

where

$$P = \sum_{\text{rows}} \sum_{\text{perm.}} p$$

is the sum over all permutation operators p for each separate row which effectively symmetrizes each row of the tableau independently and

$$Q = \sum_{\text{columns}} \sum_{\text{perm.}} \text{sgn}(q) q$$

is the corresponding antisymmetrizing operator for each column. After applying Y to the tableau, every column is completely antisymmetrized, but the rows are not necessarily symmetric anymore, since the action of Q can destroy that symmetry again. Only in special cases (i.e. only one row or only one column), the resulting tensor is totally (anti)symmetric in all indices, in most cases it has mixed symmetry.

Very important tableaus are those consisting of only one row (totally symmetric) and only one column (totally antisymmetric). Note that in SU(N), a Young tableau cannot have more than N rows, since N rows correspond to N totally antisymmetric indices. Hence their values are fixed and that part of the tensor corresponds to an invariant scalar. Therefore, columns containing N rows are often ignored when drawing Young diagrams.

2 Dimensions of Young tableaux/irreducible representations

We will give here (without proof) an algorithm for determining the dimension of a Young tableau and therefore of the corresponding irreducible representation in SU(N). First, start by drawing two identical Young tableaux. In the first, write N in the top left box and then fill all the other boxes so that the number in each is one greater than that in the box to the left and one lower than that in the box above. In the second tableau, write in each box the number of boxes to the right plus the number of boxes below plus one. For the tableau given above and SU(3), this gives

$$\begin{array}{|c|c|c|c|}\hline 3 & 4 & 5 & 6 \\ \hline 2 & 3 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \Bigg/ \begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

As a final step, multiply all numbers in each diagram together and divide the results. In this case, this gives

$$\frac{2 \cdot 3^2 \cdot 4 \cdot 5 \cdot 6}{2 \cdot 3 \cdot 4 \cdot 6} = 15$$

3 Tensor products and their decompositions

Of particular interest is the decomposition of tensor products. In the case of two 1/2-spins for example this corresponds to the decomposition $\left|\frac{1}{2}, m_1\right\rangle \otimes \left|\frac{1}{2}, m_2\right\rangle = |1, m\rangle \oplus |0, 0\rangle$, in the case of particle physics it corresponds to multiplett construction out of the tensor products $q\bar{q}$ (mesons) or qqq (baryons).

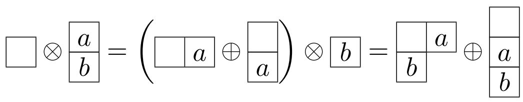
In order to describe (again without proof) the algorithm for multiplying two Young tableaux together, we first have to introduce the definition of an admissible sequence. A sequence of letters $a, b, c, \ldots$ is *admissible* at any point in the sequence no more b's than a's have occured, no more b's than c's, etc. Thus *aababccb* is admissible, while *ababccb* is not.

In order to multiply two tableaux $T_1 \otimes T_2$ together, start by writing down both tableaux and fill T_2 with letters so that all boxes in the first row contain a's, in the second row contain b's etc.

1. Enlarge T_1 with all a's from T_2 so that all created new tableaux are valid Young tableaux with (in case of SU(N)) have at most N rows and where every letter appears only once in every column (otherwise that tableau would correspond to an antisymmetrization of two symmetric indices).
2. Repeat the frist step for all other letters subsequently, making sure that each tableau consists of an admissible sequence of letters when reading from right to left and top to bottom at all times.

The resulting tableaux correspond to the Clebsch-Gordan decomposition of the tensor product.

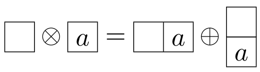
Example: Mesons. Coupling of a quark q and anti-quark $\bar{q}$ in SU(3). As mentioned above, q corresponds to a tensor ψ_a, while $\bar{q}$ corresponds to $\phi^a = \phi_{[ij]}$. Hence



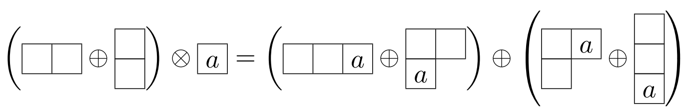
Writing this with the dimensionalities yields

$$3 \otimes \bar{3} = 8 \oplus 1$$

Example: Baryons: Coupling three quarks in SU(3). Coupling the first two to qq gives



Coupling the third gives



Writing this with the dimensionalities yields

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

References

- H.F. Jones, *Groups, Representations and Physics*, 2nd edition, Taylor & Francis.
- *SU(n)Mulitplets and Young Diagrams* from
  D.E. Groom *et al.*, *The European Physical Journal* **C15** (2000) 1 and 2001 off-year partial update for the 2002 edition available on the PDG WWW pages (URL: `http://pdg.lbl.gov/`)