Hand-in exercises in the course QCD@Colliders, HT 2006

1. Fundamentals of QCD

A. Colour SU(3):

Prove the Jacobi identity $f^{ABC}f^{CKL} + f^{ACL}f^{CKB} = f^{ACK}f^{BCL}$ for the SU(3) structure constants. (Use $[T^A, T^B] = if^{ABC}T^C$ and $(T^A)_{BC} = -if^{ABC}$.)

B. Local gauge invariance:

Show that the gauge transformations $q_a(x) \to q'_a(x) = \exp(it \cdot \theta(x))_{ab}q_b(x) \equiv \Omega(x)_{ab}q_b(x)$ and (dropping the colour index) $D_\alpha q(x) \to D'_\alpha q'(x) \equiv \Omega(x)D_\alpha q(x)$ gives the following transformation property of the gluon field

$$t \cdot \mathcal{A}_\alpha \to t \cdot \mathcal{A}'_\alpha = \Omega(x)t \cdot \mathcal{A}_\alpha \Omega^{-1}(x) + \frac{i}{g}(\partial_\alpha \Omega(x))\Omega^{-1}(x)$$

and that as a consequence the field strength tensor transforms as

$$t \cdot F_{\alpha\beta}(x) \to t \cdot F'_{\alpha\beta}(x) = \Omega(x)t \cdot F_{\alpha\beta}(x)\Omega^{-1}(x)$$

Show that a mass term for the gluon, $m^2\mathcal{A}^\alpha\mathcal{A}_\alpha$, breaks gauge invariance

C. Feynman rules:

Show that the inverse gluon propagator, $i\delta_{AB}\left[p^2 g_{\alpha\beta} - (1-\frac{1}{\lambda})p_\alpha p_\beta\right]$, cannot be inverted without the gauge-fixing term.

Show that the colour-factor of the four-gluon vertex can be written as a sum of products of three-gluon vertices (three terms with two factors). What about the Lorentz factors?

Calculate the colour factor for triple-gluon and four-gluon scattering via a quark loop. (Hint. For the fundamental representation we have $Tr(t^At^B) = \frac{1}{2}\delta^{AB}$ and from the defining relations $[t^A, t^B] = if^{ABC}t^C$ and $\{t^A, t^B\} = \frac{1}{N_C}\delta_{ab}\mathbb{1} + d^{ABC}t^C$ we have $t^At^B = \frac{1}{2N_C}\delta^{AB}\mathbb{1} + \frac{1}{2}d^{ABC}t^C + \frac{1}{2}if^{ABC}t^C$.)

Derive the Feynman rules involving heavy quarks in Fig 1.4 of the book QCD and Colliders from the corresponding general ones and the properties of the projection operator (1.66).